Application of Eigenstructure Assignment to Flight Control Design: Some Extensions

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The eigenstructure assignment flight control design methodology is extended to include dynamic compensator synthesis and damping ratio sensitivity reduction. Dynamic compensators may be designed via eigenstructure assignment by utilizing a composite system structure. The success of this design methodology depends upon proper choice of the desired eigenvectors. Sensitivity measures are developed that relate the perturbation of the damping ratio to perturbations in the stability derivatives. A damping ratio sensitivity plot is introduced that allows the damping ratio sensitivity to be reduced without altering the nominal damping ratio. Examples of the lateral dynamics of an L-1011 aircraft are presented to illustrate the design methods.

Introduction

EIGENSTRUCTURE assignment has been shown to be a useful tool for flight control system design. This method allows the designer to directly satisfy damping, settling time, and decoupling specifications by directly choosing the eigenvalues and eigenvectors. Andry et al.[1] have applied eigenstructure assignment using constrained output feedback to the design of a stability augmentation system for the lateral dynamics of an L-1011 aircraft. Sobel and Shapiro[2,3] have applied eigenstructure assignment to the design of specialized task-tailored flight control laws for advanced fighter aircraft.

The control laws of Refs. 1–3 are limited to constant-gain output feedback. In the event that the aircraft cannot be adequately controlled with constant-gain output feedback, a reduced-order observer may be utilized. However, it may be more desirable to incorporate dynamic compensation rather than an explicit state observer. Therefore, one contribution of this paper is to extend the eigenstructure assignment flight control design methodology to include the design of dynamic compensators. We shall utilize a composite system structure originally proposed by Johnson and Athans[4] and show that the success of this design methodology depends upon proper choice of the desired eigenvectors. We remark that Porter[14] has considered dynamic compensator synthesis utilizing eigenstructure assignment. However, the approach presented here differs from that shown in Ref. 14 regarding the choice of closed-loop eigenvectors.

A secondary objective in flight control design is to obtain closed-loop eigenvalues that are less sensitive to parameter uncertainty or parameter variation. We extend the results of Gilbert[5] to develop sensitivity measures that describe the changes in the damping of complex-conjugate eigenvalues due to changes in the entries of the closed-loop system matrix. The damping sensitivity may be of interest to flight control designers because the damping is directly related to physical quantities such as maximum percent overshoot and settling time. A flight control design procedure is presented that uses eigenstructure assignment with damping sensitivity reduction.

Two examples are presented that utilize the seventh-order lateral dynamics of the L-1011 aircraft to illustrate the design methods. In the first example, we measure only washed-out yaw rate, roll rate, and bank angle. However, by utilizing a dynamic compensator, we can assign both the roll mode and dutch-roll mode eigenvalues and parts of their corresponding eigenvectors. In the second example, we utilize a gradient search to reduce the damping sensitivity. Since this technique reduces the damping, a new method that utilizes the damping sensitivity plot is introduced. This new technique results in reduced damping sensitivity without changing the nominal damping.

Eigenstructure Assignment

Consider the linear time invariant system described by

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx \tag{2}$$

where $x \in R^n$, $u \in R^m$, and $y \in R^r$.

We shall assume that

$$\text{rank}[B] = m \tag{3}$$

and

$$\text{rank}[C] = r \tag{4}$$

Under the above assumptions, the feedback problem can be stated as follows: Given a set of desired eigenvalues $\{\lambda_i^d\}$, $i = 1, 2, \ldots, r$ and a corresponding set of desired eigenvectors, $\{v_i^d\}$, $i = 1, 2, \ldots, r$, find a real $m \times r$ matrix F such that the eigenvalues of $A + BFC$ contain $\{\lambda_i^d\}$ as a subset and the corresponding eigenvectors of $A + BFC$ are close to the respective members of the set $\{v_i^d\}$.

The following theorem, due to Srinathkumar,[6] describes the number of eigenvalues and eigenvector entries that can be exactly assigned.

Theorem: Given the controllable and observable system described in Eqs. (1) and (2) and the assumptions that the matrices B and C are full rank, then max (m, r) closed-loop eigenvalues can be assigned and max (m, r) eigenvectors can be partially assigned with min (m, r) entries in each vector arbitrarily chosen using constant-gain output feedback.

In general, we may desire to exercise some control over more than min (m, r) entries in a particular eigenvector. Therefore, we

Received Sept. 9, 1985; revision received March 4, 1986.

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shall now discuss the problems of first characterizing desired eigenvectors v_i^d that can be assigned as closed-loop eigenvectors and of then determining the best possible set of achievable eigenvectors in case a desired eigenvector v_i^d is not achievable.

The solution to these problems was shown in Ref. 1 and begins with the closed-loop system

$$\dot{x}(t) = (A + BFC)x(t) \tag{5}$$

For an eigenvalue/eigenvector pair, λ_i and v_i

$$(A + BFC)v_i = \lambda_i v_i \tag{6}$$

or

$$v_i = (\lambda_i I - A)^{-1}BFCv_i \tag{7}$$

Define the m vector m_i as

$$m_i = FCv_i \tag{8}$$

Then Eq. (7) becomes

$$v_i = (\lambda_i I - A)^{-1}Bm_i \tag{9}$$

To ensure that the inverse in Eq. (9) exists, we shall assume that controllable eigenvalues are moved if the corresponding eigenvectors are to be altered. This assumption, together with the complete controllability assumption, may be removed by using results derived by Liebst and Garrard.[7,8]

The importance of Eq. (9) is the need for the eigenvector v_i to be in the subspace spanned by the columns of $(\lambda_i I - A)^{-1}B$. This subspace is of dimension m, which is the number of independent control variables. Therefore, the number of control variables determines the dimension of the subspace in which the achievable eigenvectors must reside. The orientation of the subspace is determined by the open-loop parameters described by A and B and the desired closed-loop eigenvalue λ_i. We conclude that if we choose an eigenvector v_i that lies precisely in the subspace spanned by the columns of $(\lambda_i I - A)^{-1}B$, it will be achieved exactly.

In general, however, a desired eigenvector v_i^d will not reside in the prescribed subspace and, hence, cannot be achieved. Instead, a "best possible" choice for an achievable eigenvector is made. This "best possible" eigenvector is the projection of v_i^d onto the subspace spanned by the columns of $(\lambda_i I - A)^{-1}B$.

To further complicate the situation, complete specification of v_i^d is neither required nor known in most practical situations. When the designer is interested in only certain elements of the eigenvector, we assume that the desired eigenvector has a structure given by

$$v_i^d = [v_{i1}, x, x, x, x, v_{ij}, x, x, x, v_{in}]^T$$

where v_{ij} are the designer specified components and x the unspecified components. We define a reordering operator $\{\ \}^{R_i}$ as follows[9]:

$$\{v_i^d\}^{R_i} = \begin{bmatrix} \ell_i^d \\ d_i \end{bmatrix}$$

where ℓ_i^d is the vector of specified components of v_i^d and d_i is the vector of unspecified components of v_i^d.

We begin the computation of an achievable eigenvector v_i^a by defining

$$L_i = (\lambda_i I - A)^{-1}B \tag{10}$$

An achievable eigenvector must reside in the required subspace and, hence,

$$v_i^a = L_i z_i \tag{11}$$

We reorder the rows of L_i to conform with the reordered components of v_i^d. Thus, as shown in Ref. 9, we have

$$\{L_i\}^{R_i} = \begin{bmatrix} \tilde{L}_i \\ D_i \end{bmatrix} \tag{12}$$

To find the value of z_i corresponding to the projection of ℓ_i^d onto the "achievability subspace," we choose z_i, which minimizes

$$J = \|\ell_i^d - \ell_i^a\|^2 = \|\ell_i^d - \tilde{L}_i z_i\|^2 \tag{13}$$

We set the first derivative of J with respect to z_i equal to zero and when dim $(\ell_i^d) \geq m$, we obtain

$$z_i = (\tilde{L}_i^T \tilde{L}_i)^{-1}\tilde{L}_i^T \ell_i^d \tag{14}$$

If dim $(\ell_i^d) < m$, then the solution is nonunique. In this case, the minimum norm solution is given by

$$z_i = \tilde{L}_i^T(\tilde{L}_i \tilde{L}_i^T)^{-1}\ell_i^d \tag{15}$$

Then, in either case, the achievable eigenvector is given by

$$v_i^a = L_i z_i \tag{16}$$

As shown in Ref. 1, the feedback gain matrix is computed by using a transformation to obtain a system equivalent to Eqs. (1) and (2). This transformed system is described by $(\tilde{A}, \tilde{B}, \tilde{C})$ where

$$\tilde{A} = T^{-1}AT \tag{17}$$

$$\tilde{B} = T^{-1}B = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \tag{18}$$

$$\tilde{C} = CT \tag{19}$$

Under this transformation,

$$x = T\tilde{x} \tag{20}$$

$$\lambda_i = \tilde{\lambda}_i \tag{21}$$

$$v_i^a = T\tilde{v}_i^a \tag{22}$$

We partition $\tilde{v}_i^a$ and $\tilde{A}$ conformally with $\tilde{B}$ to obtain

$$\tilde{v}_i^a = \begin{bmatrix} \tilde{s}_i \\ \hline \tilde{w}_i \end{bmatrix} \tag{23}$$

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 \\ \hline \tilde{A}_2 \end{bmatrix} \tag{24}$$

Then, as shown in Ref. 1, the feedback gain matrix is computed by using

$$F = (\tilde{S} - \tilde{A}_1\tilde{V})(\tilde{C}\tilde{V})^{-1} \tag{25}$$

where

$$\tilde{S} = [\lambda_1\tilde{s}_1, \lambda_2\tilde{s}_2, \ldots, \lambda_r\tilde{s}_r] \tag{26}$$

and

$$\tilde{V} = [\tilde{v}_1^a, \tilde{v}_2^a, \ldots, \tilde{v}_r^a] \tag{27}$$

The feedback gain matrix given by Eq. (25) feeds back every output to every input. We now consider the problem of constraining certain elements of F to be zero. By suppressing certain gains to zero, the designer reduces controller complexity and increases reliability. As shown in Ref. 1, we rearrange Eq. (25) to obtain

$$F\tilde{C}\tilde{V} = \tilde{S} - \tilde{A}_1\tilde{V} \tag{28}$$

By definition we let

$$\Omega = \tilde{C}\tilde{V} \tag{29}$$

and

$$\Psi = \tilde{S} - \tilde{A}_1\tilde{V} \tag{30}$$

As shown in detail in Ref. 1, each row of feedback gains (f_i) can be computed independently of all other rows. Mathematically,

$$f_i = \Psi_i\Omega^{-1} \tag{31}$$

If we were to constrain f_{ij} to be zero, then we delete f_{ij} from f_i^T and delete the jth column of Ω^T. We now solve the reduced problem,

$$\tilde{\Omega}^T\tilde{f}_i^T = \Psi_i^T \tag{32}$$

where $\tilde{\Omega}^T$ is the matrix Ω^T with its jth column deleted and $\tilde{f}_i^T$ is the vector f_i^T with its jth element deleted.

Equation (32) may be solved for $\tilde{f}_i$, the remaining active gains in the ith row, as shown in Ref. 1. If more than one gain in a row of F is to be set to zero, then Eq. (32) should be appropriately modified.

We remark that Eq. (32) represents an overdetermined system of equations and only a least squares solution may be obtained. Therefore, in the constrained output feedback case, we cannot guarantee eigenvalue/eigenvector assignment for the max (m, r) eigenvalue/eigenvector pairs to be assigned. However, we may obtain a good engineering solution.

We conclude the discussion with a comment about the closed-loop system stability. Unfortunately, it is not yet possible to ensure

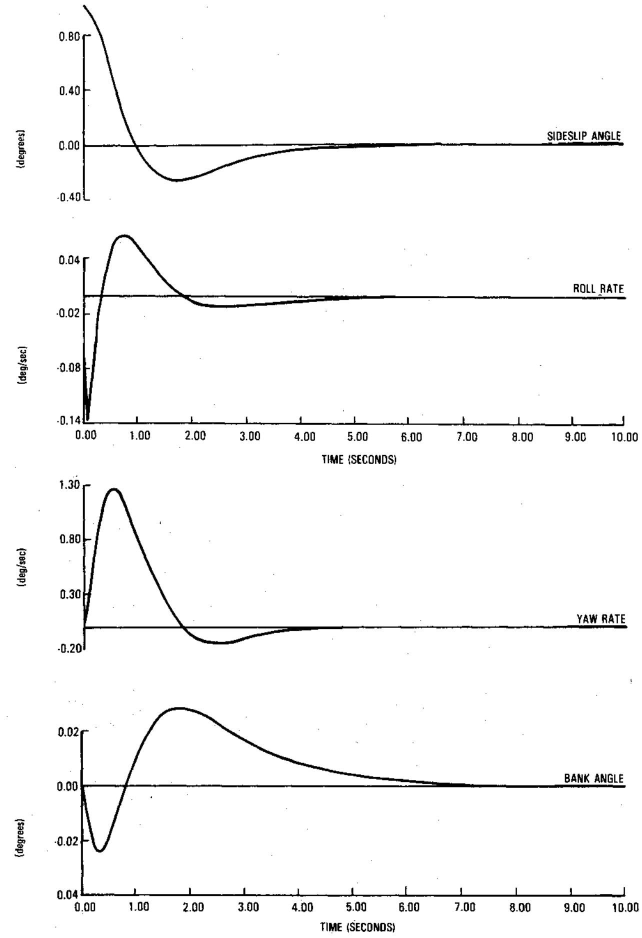

Fig. 1 Constant gain aircraft responses.

Table 1 Comparison of constant-gain feedback and dynamic compensation

	Desired Eigenvalves	Achievable Eigenvalues	Desired Eigenvectors: Dutch roll mode	Roll mode	Compensator mode	
CONSTANT GAIN FEEDBACK	$\lambda_{dr} = -1.5 \pm j\,1.5$	$\lambda_{dr} = -1.5 + j\,1.5$	X ± j X	X ± j X		δ_r
	$\lambda_{roll} = -2.0 \pm j\,1.0$	$\lambda_{roll} = -2.0 \pm j\,1.0$	X ± j X	X ± j X		δ_a
		$\lambda_{act} = -22.01$	0 ± j 0	1 ± j X		ϕ
		$\lambda_{act} = -17.05$	1 ± j X	0 ± j 0		r
		$\lambda_{filt} = -0.699$	0 ± j 0	X ± j 1		p
			X ± j 1	0 ± j 0		β
			X ± j X	0 ± j 0		X_7
FIRST ORDER COMPENSATOR	$\lambda_{dr} = -1.5 \pm j\,1.5$	$\lambda_{dr} = -1.5 \pm j\,1.5$	X ± j X	X ± j X		δ_r
	$\lambda_{roll} = -2.0 \pm j\,1.0$	$\lambda_{roll} = -2.0 \pm j\,1.0$	X ± j X	X ± j X		δ_a
		$\lambda_{act} = -22.12$	0 ± j 0	1 ± j X		ϕ
		$\lambda_{act} = -18.72$	1 ± j X	0 ± j 0		r
		$\lambda_{7,8} = -0.402 + j\,0.423$	0 ± j 0	X ± j 1		p
			X ± j 1	0 ± j 0		β
			X ± j X	0 ± j 0		X_7
			1 ± j X	1 ± j X		Z_1
SECOND ORDER COMPENSATOR	$\lambda_{dr} = -1.5 + j\,1.5$	$\lambda_{dr} = -1.5 + j\,1.5$	X ± j X	X ± j X	X	δ_r
	$\lambda_{roll} = -2.0 \pm j\,1.0$	$\lambda_{roll} = -2.0 \pm j\,1.0$	X ± j X	X ± j X	X	δ_a
	$\lambda_{comp} = -1.5$	$\lambda_{act} = -18.05 \pm j\,3.08$	0 ± j 0	1 ± j X	X	ϕ
		$\lambda_{filt} = -0.7862$	1 ± j X	0 ± j 0	X	r
		$\lambda_8 = -1.5$	0 ± j 0	X ± j 1	X	p
		$\lambda_9 = -3.48$	X ± j 1	0 ± j 0	X	β
			X ± j X	0 ± j 0	X	X_7
			1 ± j X	1 ± j X	1	Z_1
			X ± j 1	X ± j 1	1	Z_2

Table 2 Comparison of achievable eigenvectors

CONSTANT GAIN FEEDBACK

	DUTCH ROLL MODE			ROLL MODE			ACTUATOR	ACTUATOR	FILTER
δ_r	-0.723		6.35	0.309		0.138	-12.7	-16.2	-3.41
δ_a	4.47		1.59	-6.05		-2.16	-0.725	-12.2	4.73
ϕ	4×10^{-6}		8×10^{-6}	1.38		-1.85	-0.0129	0.295	0.244
r	-1.26	±j	1.96	0.01	±j	-0.011	-0.561	-0.0235	-0.799
p	-1×10^{-5}		-1×10^{-5}	-4.6		2.33	0.220	-0.649	-0.170
β	-1.07		0.162	-8×10^{-3}		0.035	-0.179	-9×10^{-4}	-1.27
x_7	0.647		-0.012	-5×10^{-4}		4×10^{-3}	0.0169	5×10^{-4}	2.02

FIRST ORDER COMPENSATOR

	DUTCH ROLL MODE			ROLL MODE			ACTUATOR	ACTUATOR	COMPENSATOR/FILTER		
δ_r	3.16		-3.77	-0.104		-0.125	-12.7	0.318	4.14		-3.79
δ_a	-2.19		-2.92	1.78		2.52	9.73	12.2	-9.33		6.89
ϕ	-1×10^{-5}		-2×10^{-6}	0.680		-0.874	-0.046	-0.029	-2.09		3.32
r	1.64	±j	-0.726	3×10^{-3}	±j	-6×10^{-3}	-0.489	0.028	-0.012	±j	-1.29
p	2×10^{-5}		-1×10^{-5}	-0.486		2.43	0.860	0.643	-0.560		-2.22
β	0.755		0.351	-0.015		8×10^{-3}	-0.013	1×10^{-3}	2.16		-1.49
x_7	-0.420		-0.267	-2×10^{-3}		8×10^{-4}	0.013	-6×10^{-4}	-14.5		-0.322
z_1	0.017		-0.763	0.681		-0.875	-0.029	-0.03	-1.69		2.82

SECOND ORDER COMPENSATOR

	DUTCH ROLL MODE			ROLL MODE			ACTUATOR MODES			COMPENSATOR	COMPENSATOR	FILTER
δ_r	-2.71		-1.48	0.197		-0.107	-7.37		-7.53	7×10^{-5}	0.555	6.13
δ_a	-1.53		1.71	-3.54		2.24	4.21		-10.4	-8×10^{-5}	0.683	-8.13
ϕ	-5×10^{-6}		-2×10^{-6}	-0.399		-1.45	-0.031		0.019	-4×10^{-5}	-0.083	-1.39
r	-0.651	±j	-0.925	-2×10^{-3}	±j	-0.011	-0.23	±j	-0.371	-2×10^{-5}	0.116	1.52
p	1×10^{-5}		-2×10^{-6}	-0.651		3.30	0.504		-0.440	6×10^{-5}	0.289	1.08
β	0.124		-0.510	-0.0146		0.018	-2×10^{-3}		-0.012	2×10^{-5}	0.032	2.18
x_7	-0.113		0.292	2×10^{-3}		2×10^{-3}	4×10^{-3}		0.011	-3×10^{-5}	-0.019	-2.69
z_1	-0.474		0.083	-0.398		-1.45	-0.018		0.025	2.86	-0.107	2.02
z_2	-0.083		-0.473	-1.45		0.399	0.042		-0.039	2.86	0.05	3.57

that stable open-loop eigenvalues do not move into the right half of the complex plane when an eigenstructure assignment output feedback controller is utilized. This is still an open area for further research. However, for aircraft flight control systems, the closed-loop stability requirement is neither necessary nor sufficient. For example, some modes, such as the dutch-roll mode, are required to meet minimum frequency and damping specifications as described in Ref. 10. For these modes, stability alone is not sufficient. Other modes, such as the spiral mode, may be unstable provided that the time to double amplitude is sufficiently large. For these modes, stability may not be necessary. Thus, a possible area for further research might be to determine when max (m, r) eigenvalues can be assigned with the other eigenvalues remaining in specified regions in the complex plane. Some of these regions might include parts of both the left and right halves of the complex plane.

Dynamic Compensators

We shall generalize the eigenstructure flight control design methodology with constant gain feedback to include the design of low-order dynamic compensators of any given order ℓ, $0 \leq \ell \leq n - r$. Recall that n and r are the dimensions of the aircraft state and measurement vectors, respectively. Consider the linear time invariant plant described by Eqs. (1) and (2) with a linear time invariant dynamical controller specified by

$$\dot{z} = Dz + Ey \tag{33}$$

$$u = Fz + Gy \tag{34}$$

where the controller state vector z is of dimension ℓ, $0 \leq \ell \leq n - r$.

It is convenient to model the plant and compensator by the composite system originally proposed by Johnson and Athans.[4] Thus, define

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}\bar{u} \\ \bar{y} &= \bar{C}\bar{x} \\ \bar{u} &= \bar{F}\bar{y} \end{aligned} \tag{35}$$

where

$$\bar{x} = \begin{bmatrix} x \\ \hline z \end{bmatrix}, \quad \bar{A} = \left[\begin{array}{c|c} A & O \\ \hline O & O \end{array}\right], \quad \bar{B} = \left[\begin{array}{c|c} B & O \\ \hline O & I \end{array}\right]$$

$$\bar{C} = \left[\begin{array}{c|c} C & O \\ \hline O & I \end{array}\right], \quad \bar{F} = \left[\begin{array}{c|c} G & F \\ \hline E & D \end{array}\right]$$

Furthermore, the eigenvectors of the composite system may be described by

$$v_i = \begin{bmatrix} v_i(x) \\ \hline v_i(z) \end{bmatrix} \tag{36}$$

where $v_i(x)$ is the ith subeigenvector corresponding to the plant and $v_i(z)$ the ith subeigenvector corresponding to the compensator.

The dynamic compensator design problem may be stated as follows. Given a set of desired plant eigenvalues $\{\lambda_i^d\}$, $i = 1, 2, \ldots, r + \ell$ and a corresponding set of desired plant subeigenvectors $v_i^d(x)$, $i = 1, 2, \ldots, r + \ell$, find real matrices D $(\ell \times \ell)$, E $(\ell \times r)$, F $(m \times \ell)$, and G $(m \times r)$ such that the eigenvalues of $A + BFC$ contain $\{\lambda_i^d\}$ as a subset and the corresponding subeigenvectors $\{v_i(x)\}$ are close to the respective members of the set $\{v_i^d(x)\}$.

For example, consider the lateral dynamics of an aircraft augmented with actuator dynamics and a washout filter on yaw rate. The state vector is given by

$$x = \begin{bmatrix} \delta_r \\ \delta_a \\ \phi \\ r \\ p \\ \beta \\ x_7 \end{bmatrix} \begin{matrix} \text{rudder deflection} \\ \text{aileron deflection} \\ \text{bank angle} \\ \text{yaw rate} \\ \text{roll rate} \\ \text{sideslip angle} \\ \text{washout filter state} \end{matrix} \tag{37}$$

and the input vector is given by

$$u = \begin{bmatrix} \delta_{rc} \\ \delta_{ac} \end{bmatrix} \begin{matrix} \text{rudder command} \\ \text{aileron command} \end{matrix} \tag{38}$$

If $y = [r_{\mathrm{wo}}, p, \beta, \phi]^T$ is measurable (where r_{wo} is washed-out yaw rate), then the designer can specify both the dutch-roll mode and roll mode eigenvalues. The designer might also specify three elements of the dutch-roll eigenvectors and four elements of the roll mode eigenvectors, as shown in Ref. 1, and achievable eigenvectors will be computed. Now suppose that the measurement vector is given by $y = [r_{\mathrm{wo}}, p, \phi]^T$, but the designer is still required to assign both the dutch-roll and roll mode eigenvalues. Using the results of this section, we might utilize a first-order compensator with state z_1. The composite system has state vector $\bar{x} = [\delta_r, \delta_a, \phi, r, p, \beta, x_7, z_1]^T$ and measurement vector given by $\bar{y} = [r_{\mathrm{wo}}, p, \phi, z_1]^T$. Thus, as before, the designer might choose to specify three or four elements of $v_i(x)$, $i = 1, 2, 3, 4$, which are elements of the dutch-roll mode and roll mode eigenvectors corresponding to the original aircraft state variables. Again, achievable eigenvectors will be computed.

We might ask whether the eigenvalue/eigenvector specifications are identical for both of the proposed problems. Certainly, the eigenvalue specifications and the corresponding $v_i^d(x)$ subeigenvector specifications may be identical. However, the $v_i^d(z)$ subeigenvectors must now be properly specified. Otherwise, the modal matrix for the composite system may become ill-conditioned or even numerically singular.

Finally, we remark that if the first-order compensator does not yield acceptable performance, then the designer might try a higher-order compensator. In the case of a second-order compensator, the composite system has state vector $\bar{x} = [\delta_r, \delta_a, \phi, r, p, \beta, x_7, z_1, z_2]^T$ and the measurement vector is given by $\bar{y} = [r_{\mathrm{wo}}, p, \phi, z_1, z_2]^T$. In this case, the designer can also specify one of the compensator eigenvalues and some elements of its corresponding eigenvector.

We consider the linearized lateral dynamical equations of the L-1011 aircraft at a cruise flight condition. The state and control vectors are given by Eqs. (37) and (38), respectively. The stability and control derivatives, which form the matrices A and B in Eq. (1), are obtained from Ref. 1 where a constant-gain output feedback control law was designed. This control law is described by

$$\begin{bmatrix} \delta_r \\ \delta_a \end{bmatrix} = \begin{bmatrix} -3.35 & 0.159 & 4.88 & 0.380 \\ -1.42 & -2.38 & 6.36 & -3.80 \end{bmatrix} \begin{bmatrix} r_{\mathrm{wo}} \\ p \\ \beta \\ \phi \end{bmatrix} \tag{39}$$

The desired eigenvalues, achievable eigenvalues, and desired eigenvectors are shown in Table 1. The achievable eigenvectors are shown in Table 2 where we observe the excellent decoupling between the dutch-roll and roll modes. The time responses are shown in Fig. 1. We remark that the closed-loop system has a dutch-roll damping ratio $\zeta_{\mathrm{dr}} \approx 0.707$ and a bank angle to sideslip ratio $|\phi/\beta| \approx 0.025$. These should be compared to the open-loop values $\zeta_{\mathrm{dr}} \approx 0.07$ and $|\phi/\beta| \approx 2.5$.

Now consider the case when only the washed-out yaw rate, roll rate, and bank angle are measured. We form the composite system described by Eq. (35) by appending a first-order compensator to the aircraft dynamics. We specify the roll mode and dutch-roll mode eigenvalues to be the same as in the constant-gain feedback problem. The compensator pole is not specified, but it is allowed to be chosen by the eigenstructure assignment algorithm in order to obtain eigenvalue and eigenvector assignment for the aircraft modes. Furthermore, since we only have three sensors plus one compensator state, the eigenstructure assignment algorithm will allow us to specify only four closed-loop eigenvalues. In general, if the compensator eigenvalues were of significant importance to the designer, then they might be chosen by constraining the entries in the submatrix D of the composite feedback gain matrix F. However, the comments regarding constrained output feedback, which follow Eq. (32), will apply. Returning to the design, we specify the desired plant subeigenvectors $v_i^d(x)$ to be the same as the desired eigenvectors in the constant-gain feedback problem. The desired compensator subeigenvectors are chosen such that both the dutch-roll and roll modes participate in the compensator state solution. The control law is described by

$$\begin{bmatrix} \delta_r \\ \delta_a \end{bmatrix} = \begin{bmatrix} 1.53 & -0.128 & -3.57 \\ -0.951 & 2.42 & 3.98 \end{bmatrix} \begin{bmatrix} r_{\mathrm{wo}} \\ p \\ \phi \end{bmatrix} + \begin{bmatrix} 3.34 \\ 4.35 \end{bmatrix} z_1 \tag{40}$$

$$\dot{z}_1 = -1.87 z_1 + 0.558 r_{\mathrm{wo}} + 1.0 p + 1.87 \phi \tag{41}$$

The desired eigenvalues, achievable eigenvalues, and desired eigenvectors are shown in Table 1. The achievable eigenvectors are shown in Table 2. We remark that the separation principle for observers does not apply to the dynamic compensator described by Eqs. (33) and (34). We observe that the closed-loop system has a pair of complex conjugate eigenvalues $\lambda_{7,8} = -0.402 \pm j0.423$ that may tend to slow the aircraft responses. The time responses are shown in Fig. 2 from which we conclude that the aircraft responses are slow and may be unacceptable.

We now consider the design of a second-order compensator. The augmented output vector is given by $\bar{y} = [r_{\mathrm{wo}}, p, \phi, z_1, z_2]^T$ and we may now specify closed-loop eigenvalues together with some elements of their corresponding eigenvectors. We specify the dutch-roll and roll mode eigenvalues to be the same as in the constant-gain feedback case. We specify the one assignable compensator eigenvalue to be $\lambda^d = -1.5$. This choice was made somewhat arbitrarily, although we tried to choose it such that the compensator would not slow down the aircraft responses. However, more research is required in order to understand the implications of choosing the assignable compensator eigenvalues. The desired plant subeigenvectors for the roll and dutch-roll modes

Table 3 Control gains and sensitivities

	FEEDBACK GAIN MATRIX					DAMPING SENSITIVITY w.r.t. N'_β
	r_{wo}	p	β	ϕ		
INITIAL DESIGN	-3.35	0.159	4.88	0.380	δ_r	$(6.95 \times 10^{-3})^{1/2}$
	-1.42	-2.38	6.36	-3.80	δ_a	
GRADIENT DESIGN	-3.70	0.201	11.5	0.561		$(6.98 \times 10^{-4})^{1/2}$
	-1.48	-2.34	8.28	-3.70		
DSP DESIGN	-6.01	0.241	17.8	0.752		$(6.98 \times 10^{-4})^{1/2}$
	-2.20	-2.35	10.4	-3.68		

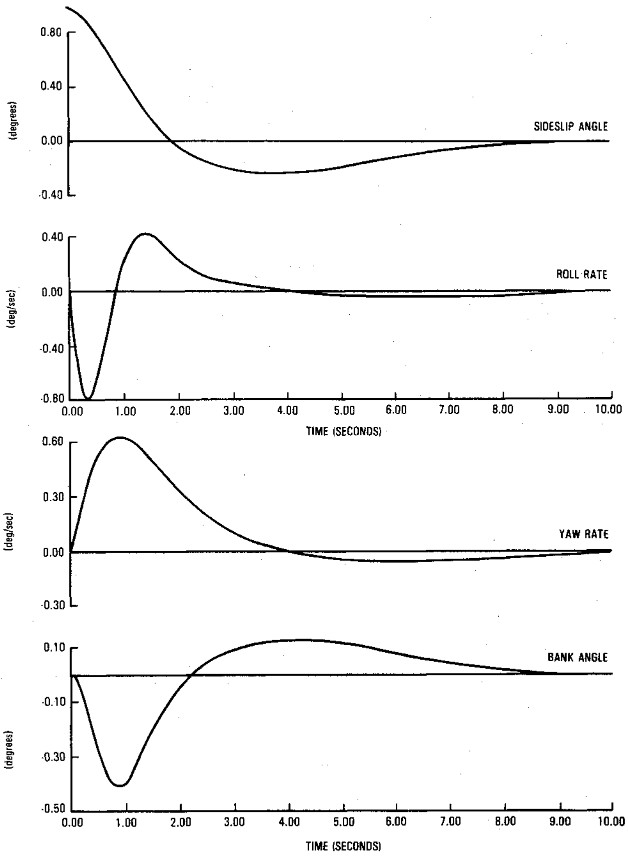

Fig. 2 First-order compensator aircraft responses.

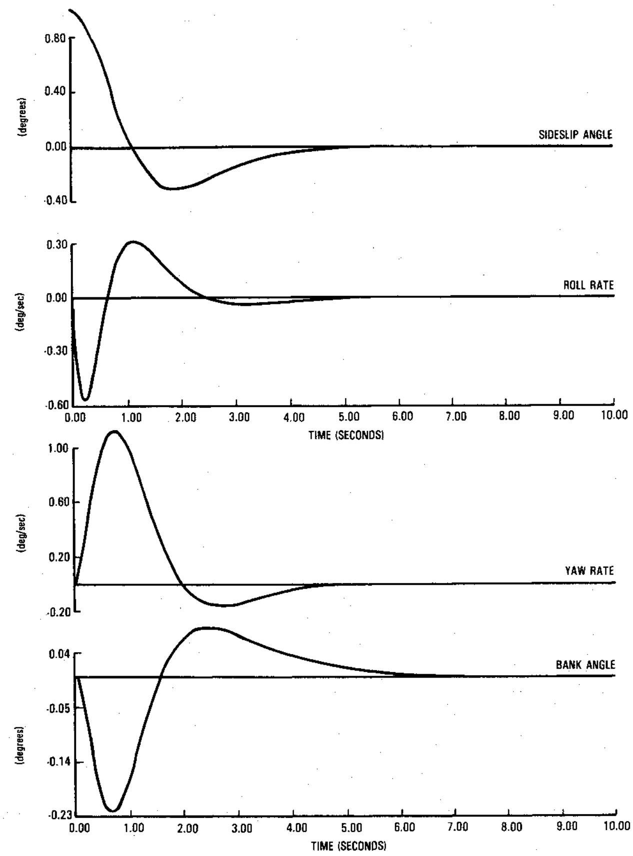

Fig. 3 Second-order compensator aircraft responses.

were specified to be the same as the desired eigenvectors in the constant-gain feedback problem. The desired compensator subeigenvectors for these two modes were chosen such that both the dutch-roll and roll modes participate in the compensator state solutions. The plant subeigenvector for the compensator eigenvector corresponding to $\lambda^d = -1.5$ is left unspecified, while the two elements of its compensator subeigenvector are chosen to be one.

The control law for the second-order compensator design is described by

$$\begin{bmatrix} \delta_r \\ \delta_a \end{bmatrix} = \begin{bmatrix} 2.53 & 2.55 & 2.44 \\ 0.348 & 5.91 & 7.47 \end{bmatrix} \begin{bmatrix} r_{wo} \\ p \\ \phi \end{bmatrix} + \begin{bmatrix} 2.68 & 2.68 \\ 3.49 & 3.49 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{42}$$

$$\dot{z}_1 = -1.80z_1 + 0.297z_2 + 0.448r_{wo} + 0.703p + 1.20\phi \tag{43}$$

$$\dot{z}_2 = -1.20z_1 - 0.297z_2 - 0.448r_{wo} - 1.70p - 3.20\phi \tag{44}$$

The desired eigenvalues, achievable eigenvalues, and desired eigenvectors are shown in Table 1. The achievable eigenvectors are shown in Table 2. We remark that the compensator eigenvalues are -1.5 and -0.6, but the closed-loop eigenvalues include -1.5 and -3.48, which again demonstrates the lack of a separation property. The time responses are shown in Fig. 3 from which we observe that the aircraft exhibits acceptable damping and settling time. In addition, the ratio $|\phi/\beta| \approx 0.25$, which is an order of magnitude less than the open-loop case. However, it is substantially larger than the constant-gain design using sideslip angle measurement.

Next, we consider the multivariable gain and phase margins for both the constant-gain design and the second-order compensator design. Suppose that the modeling errors may be described by the matrix L given by

$$L = \text{Diag}\,(\ell_1 e^{j\phi_1}, \ell_2 e^{j\phi_2}, \ldots, \ell_m e^{j\phi_m})$$

Then, as shown by Lehtomaki,[11] multivariable gain and phase margins at the input port may be defined.

Let $\sigma_{\min}\,[I + KG(s)] > \gamma$. Then, the upward gain margin is at least as large as $1/(1 - \gamma)$ and the gain reduction margin is at least as small as $1/(1 + \gamma)$. The phase margins are at least $\pm 2SIN^{-1}(\gamma/2)$. For both designs, the plant transfer matrix is given by $G(s) = C(sI - A)^{-1}B$. For the constant-gain design $K(s) = -F$ and for the dynamic compensator

$$K(s) = -\,[F(sI - D)^{-1}E + G]$$

For the constant-gain design, $\gamma = 0.82$, which yields

$$GM \in [-5.2\text{dB}, 14.9\text{dB}]$$

$$PM \in [-48.4 \text{ deg}, +48.4 \text{ deg}]$$

and for the second-order compensator design, $\gamma = 0.57$, which yields

$$GM \in [-3.9\text{dB}, 7.33\text{dB}]$$

$$PM \in [-33.1 \text{ deg}, +33.1\text{dB}]$$

We observe that the gain and phase margins are less for the compensator based design than for the constant gain design. Nevertheless, the reduction in margins does not appear to be catastrophic, especially since these margins are known to be conservative. However, an area for future research might be to incorporate robustness improvement methods, such as the method proposed by Newsom and Mukhopadhyay[12] into the eigenstructure assignment design methodology. The objective would not be robustness optimization, but rather robustness improvement while maintaining the closed-loop eigenstructure "sufficiently" close to the desired eigenstructure.

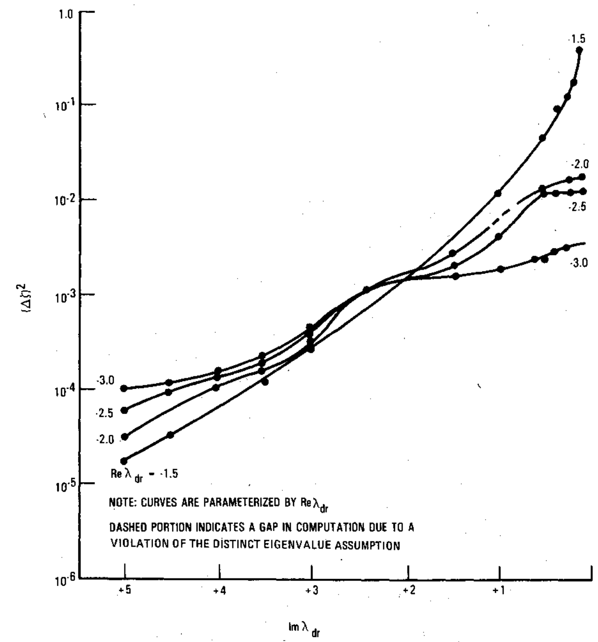

Fig. 4 Damping sensitivity plot.

Finally, we remark that our examples are presented only to illustrate the design methodology. The design methodology is based upon linearized models and a full six degree-of-freedom nonlinear simulation would be required before a final design could be obtained.

Damping Sensitivity

A secondary objective in flight control design is to obtain closed-loop eigenvalues that are less sensitive to parameter uncertainty or parameter variation. Gilbert[5] has derived measures for the sensitivities of the real part, imaginary part, and magnitude of complex-conjugate eigenvalues with respect to perturbations in the entries of the closed-loop system matrix. However, in flight control design, we might be interested in damping sensitivity because the damping is directly related to physical quantities such as maximum percent overshoot and settling time.

In this section, we extend the results in Ref. 5 to include a sensitivity measure that describes changes in the damping ratio of complex-conjugate eigenvalues due to changes in the entries of the closed-loop system matrix. This measure is most useful when the designer knows which elements of the closed-loop system matrix are subject to uncertainty. Another result in Ref. 5 describes sensitivity measures for the real part, imaginary part, and magnitude of complex conjugate eigenvalues based upon the Euclidean matrix norm on all n^2 elements of $\partial(\cdot)/\partial a_{ij}$, where a_{ij} is the ijth element of the closed-loop system matrix. We shall extend this result to include a Euclidean norm sensitivity measure for the eigenvlaue damping ratio. These types of measures are most useful when all the elements of the closed-loop system matrix are expected to have uncertainty of comparable magnitude.

Let us suppose that the matrices A, B, and C of Eqs. (1) and (2) depend continuously on a scalar parameter α. Following the work of Raman and Calise,[13] we may describe the closed-loop system with feedback control law $u = Fy$ as follows:

$$\dot{x} = [A(\alpha) + B(\alpha)\, FC(\alpha)]\, x = A_c(\alpha)\, x \tag{45}$$

We can express the variation $dA_c(\alpha)$ as

$$dA_c(\alpha) = [\delta A + \delta BFC + BF\delta C]d\alpha = \delta A_c d\alpha \tag{46}$$

where

$$\delta A = A_\alpha|_{\alpha_0},\ \delta B = B_\alpha|_{\alpha_0} \tag{47}$$

$$\delta C = C_\alpha|_{\alpha_0}$$

Result 1: Let $\lambda = \sigma + j\omega$ be a distinct complex eigenvalue of the closed-loop system matrix A_c with damping ζ, natural frequency ω_n, eigenvector $v = r + js$, and reciprocal vector $\hat{v} = \hat{r} + j\hat{s}$. Then the perturbation in damping ($d\zeta$) due to perturbations in the closed-loop system matrix (dA_c) is given by

$$d\zeta = \frac{-(1-\zeta^2)^{1/2}}{\omega_n}[(1-\zeta^2)^{1/2}(\hat{r}^T dA_c r - \hat{s}^T dA_c s) + \zeta(\hat{s}^T dA_c r + \hat{r}^T dA_c s)] \tag{48}$$

Proof: The eigenvalue λ may be written in terms of ζ and ω_n as follows:

$$\lambda = -\zeta\omega_n + j\omega_n(1-\zeta^2)^{1/2} \tag{49}$$

Define $\lambda_R = -\zeta\omega_n$ (50)

$$\lambda_I = \omega_n(1-\zeta^2)^{1/2} \tag{51}$$

Assume $0 < \zeta < 1$. Then

$$d\lambda_R = -\zeta \cdot d\omega_n - \omega_n \cdot d\zeta \tag{52}$$

$$d\lambda_I = (1-\zeta^2)^{1/2} \cdot d\omega_n - \zeta\omega_n(1-\zeta^2)^{-1/2} \cdot d\zeta \tag{53}$$

Upon solving the simultaneous equations (52) and (53) for $d\zeta$, we obtain

$$d\zeta = \frac{-(1-\zeta^2)^{1/2}}{\omega_n}[(1-\zeta^2)^{1/2} d\lambda_R + \zeta d\lambda_I] \tag{54}$$

Gilbert's[5] results for $d\lambda_R$ and $d\lambda_I$ are

$$d\lambda_R = \hat{r}^T dA_c r - \hat{s}^T dA_c s \tag{55}$$

$$d\lambda_I = \hat{s}^T dA_c r + \hat{r}^T dA_c s \tag{56}$$

Upon substituting Eqs. (55) and (56) into Eq. (54), we obtain the desired result.

Result 2: Let the assumptions and notation of result 1 hold. Then, the Euclidean matrix norm for damping sensitivity is given by

$$\left\|\frac{\partial\zeta}{\partial a_{ij}}(A_c)\right\|^2 = \|\hat{r}\|^2 + \|\hat{s}\|^2 + \frac{1-\zeta^2}{\omega_n^2}(\|s\|^2 - 1)\,\|\zeta\hat{r} - (1-\zeta^2)^{1/2}\hat{s}\|^2 \tag{57}$$

This result is an extension of Eq. (3.12) in Ref. 5 and may be derived in a similar manner. We shall discuss the idea behind extending the proof of Ref. 5 to our result. Using a development identical to the proof of result 1 we may obtain

$$\frac{\partial\zeta}{\partial a_{ij}}(A_c) = \epsilon_1\left[\epsilon_2\frac{\partial \mathrm{Re}\lambda}{\partial a_{ij}}(A_c) + \epsilon_3\frac{\partial \mathrm{Im}\lambda}{\partial a_{ij}}(A_c)\right] \tag{58}$$

where $\epsilon_1 = -(1-\zeta^2)^{1/2}/\omega_n$, $\epsilon_2 = (1-\zeta^2)^{1/2}$, and $\epsilon_3 = \zeta$. We note that Eq. (58) is of the same form as the expression for $\partial|\lambda|/\partial a_{ij}(A_c)$ given by Eq. (3.8) in Ref. 5 where ϵ_1, ϵ_2, ϵ_3 were defined as

$$\epsilon_1 = |\mathrm{Re}\lambda + j\,\mathrm{Im}\lambda|^{-1},\ \epsilon_2 = \mathrm{Re}\,\lambda,\ \epsilon_3 = \mathrm{Im}\,\lambda$$

Hence, Eq. (57) may be obtained using a development similar to that which was used to obtain Eq. (3.12) in Ref. 5. Finally, we remark that it can be shown that the minimum value of the norm given by the square root of the right hand side of Eq. (57) is $1/\sqrt{2}$.

We now present an example to illustrate the application of the damping sensitivity to flight control design. We begin with the output feedback control law for the lateral dynamics of the L-1011 aircraft as shown in Ref. 1. One advantage of the eigenstructure assignment design in Ref. 1 is that the choice of eigenvectors results in a decoupling of the dutch-roll mode and the roll mode. As a result, the dutch-roll eigenvalues are insensitive to roll mode parameters. Here, we choose to reduce the sensitivity of the dutch-roll damping ratio with respect to N'_β, which is yawing moment due to sideslip, by perturbing the desired dutch-roll eigenvalues. The sensitivity equation is obtained by substituting the appropriate dA_c into Eq. (48).

The output feedback control law computed in Ref. 1 is shown in Tables 3 and 4 as the initial design. The desired eigenvalues were chosen to be

$$\lambda_{dr} = -1.5 \pm j1.5$$

$$\lambda_{roll} = -2.0 \pm j1.0$$

We compute the gradient given by

$$g = \frac{\Delta \text{ damping sensitivity w.r.t. } N'_\beta}{\Delta \text{ desired eigenvalue locations}}$$

(where w.r.t. = with respect to). Then we search in the negative gradient direction until the damping sensitivity is reduced by an order of magnitude. This design is shown in Tables 3 and 4 as the gradient design. Observe from Table 4 that the change in damping ratio due to changes in N'_β has been significantly reduced at the expense of a decrease in the nominal damping ratio.

To obtain a decrease in the damping ratio sensitivity without a decrease in the nominal damping, we introduce the damping sensitivity plot (DSP). The DSP for our example is shown in Fig. 4 where the damping sensitivity is plotted vs Im (λ_{dr}) with the family of curves parameterized by Re (λ_{dr}). Using the DSP, we find that an order of magnitude reduction in the damping sensitivity can be obtained by choosing the desired dutch-roll eigenvalues as

$$\lambda_{dr} = -3 \pm j3$$

which corresponds to $\zeta = 0.707$ and $\omega_n = 4.24$. Thus, we can achieve an order of magnitude reduction in the damping sensitivity without changing the nominal ζ, by increasing the nominal dutch-roll natural frequency. The nominal roll mode eigenvalues are not changed. Furthermore, the roll mode eigenvalues are insensitive to N'_β perturbations due to the decoupling incorporated in the desired eigenvectors.

Finally, in Table 5 we show the variation in dutch-roll damping ratio and natural frequency due to variations in the other stability derivatives. The values shown are for variations in each stability derivative individually from zero to twice its nominal value. We conclude that the DSP design has not significantly increased the sensitivity of the dutch-roll damping ratio with respect to perturbations in the other stability derivatives.

Table 4 Comparison of control law designs

N'_β	Initial design	Gradient design	DSP design
	$\zeta_{dr} = 0.912$	$\zeta_{dr} = 0.550$	$\zeta_{dr} = 0.753$
0.0	$(\omega_n)_{dr} = 1.45$	$(\omega_n)_{dr} = 2.98$	$(\omega_n)_{dr} = 4.01$
	$\lambda_{roll} = -2.001 \pm j\,1.000$	$\lambda_{roll} = -1.981 \pm j\,1.004$	$\lambda_{roll} = -1.998 \pm j\,1.003$
	$\zeta_{dr} = 0.708$	$\zeta_{dr} = 0.504$	$\zeta_{dr} = 0.708$
1.54 (Nominal)	$(\omega_n)_{dr} = 2.12$	$(\omega_n)_{dr} = 3.25$	$(\omega_n)_{dr} = 4.24$
	$\lambda_{roll} = -2.001 \pm j\,0.9997$	$\lambda_{roll} = -1.981 \pm j\,1.004$	$\lambda_{roll} = -1.998 \pm j\,1.003$
	$\zeta_{dr} = 0.608$	$\zeta_{dr} = 0.468$	$\zeta_{dr} = 0.670$
3.08	$(\omega_n)_{dr} = 2.51$	$(\omega_n)_{dr} = 4.45$	$(\omega_n)_{dr} = 4.45$
	$\lambda_{roll} = -2.001 \pm j\,0.9998$	$\lambda_{roll} = -1.981 \pm j\,1.004$	$\lambda_{roll} = -1.998 \pm j\,1.002$

ζ_{dr} DUTCH ROLL DAMPING

$(\omega_n)_{dr}$ DUTCH ROLL NATURAL FREQUENCY

Table 5 Dutch-roll damping variation due to stability derivative variation

	INITIAL DESIGN	DSP DESIGN
N'_r (YAWING MOMENT DUE TO YAW RATE)	$0.666 \leq \zeta_{dr} \leq 0.751$	$0.684 \leq \zeta_{dr} \leq 0.734$
	$2.12 \leq \omega_{dr} \leq 2.12$	$4.22 \leq \omega_{dr} \leq 4.26$
N'_p (YAWING MOMENT DUE TO ROLL RATE)	$0.708 \leq \zeta_{dr} \leq 0.708$	$0.708 \leq \zeta_{dr} \leq 0.708$
	$2.12 \leq \omega_{dr} \leq 2.12$	$4.24 \leq \omega_{dr} \leq 4.24$
L'_r (ROLLING MOMENT DUE TO YAW RATE)	$0.707 \leq \zeta_{dr} \leq 0.710$	$0.707 \leq \zeta_{dr} \leq 0.710$
	$2.12 \leq \omega_{dr} \leq 2.12$	$4.24 \leq \omega_{dr} \leq 4.24$
L'_p (ROLLING MOMENT DUE TO ROLL RATE)	$0.708 \leq \zeta_{dr} \leq 0.708$	$0.708 \leq \zeta_{dr} \leq 0.708$
	$2.12 \leq \omega_{dr} \leq 2.12$	$4.24 \leq \omega_{dr} \leq 4.24$
L'_β (ROLLING MOMENT DUE TO SIDESLIP)	$0.707 \leq \zeta_{dr} \leq 0.708$	$0.707 \leq \zeta_{dr} \leq 0.710$
	$2.07 \leq \omega_{dr} \leq 2.17$	$4.20 \leq \omega_{dr} \leq 4.28$
Y_β (SIDEFORCE DUE TO SIDESLIP)	$0.703 \leq \zeta_{dr} \leq 0.712$	$0.706 \leq \zeta_{dr} \leq 0.711$
	$1.98 \leq \omega_{dr} \leq 2.25$	$4.13 \leq \omega_{dr} \leq 4.35$

NOTE: DAMPING RATIOS AND NATURAL FREQUENCIES SHOWN FOR PARAMETER VARIATION BETWEEN ZERO AND TWICE THE NOMINAL VALUE.

Conclusions

We have extended the eigenstructure assignment flight control design methodology to include dynamic compensator synthesis and damping ratio sensitivity reduction. Dynamic compensators may be designed by using a composite system structure. The designer can often obtain a good engineering solution by using this method, although closed-loop stability is not guaranteed. Sensitivity measures were derived which describe changes in eigenvalue damping ratios due to changes in the entries of the closed-loop system matrix. A design method was presented that results in reduced damping ratio sensitivity without changing the value of the nominal damping ratio.

Acknowledgments

The authors thank Mr. Howard Lee and Dr. David Rodabaugh, both with Lockheed California Company, for valuable discussions during the course of this work.

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